Chapter 2

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Wave Equation

Continuous PDE: $xmin \leq x \leq xmax$, with $a =$ constant

$$\frac{\partial u(x,t)}{\partial t} + a\frac{\partial u(x,t)}{\partial x} = 0 \tag{1}$$

1. PDE Theory requires an Initial Condition (IC) and Boundary Conditions (BC)
2. IC: $u(x,0) = g(x)$, an arbitrary function of x, must satisfy BC
3. BC: The first order PDE in x requires only one BC, satisfying IC
   (a) If $a \geq 0$, then $u(xmin,t) = l(t)$
   (b) If $a < 0$, then $u(xmax,t) = r(t)$

Discussion of BC: Non-Periodic

1. Scalar quantity u is given on one boundary, corresponding to a wave entering the domain thru this "inflow" boundary.
   (a) No boundary condition is specified at the opposite side, the "outflow" boundary.
   (b) This is consistent in terms of the well-posed-ness of a first-order PDE.
   (c) Hence the wave leaves the domain through the outflow boundary without distortion or reflection.
   (d) Note that the left-hand boundary is the inflow boundary when a is positive, while the right-hand boundary is the inflow boundary when a is negative.

Discussion of BC: Periodic

1. The flow being simulated is periodic.
   (a) At any given time, what enters on one side of the domain must be the same as that which is leaving on the other.
   (b) This is referred to as the *biconvection* problem.
   (c) It is the simplest to study and serves to illustrate many of the basic properties of numerical methods applied to problems involving convection, without special consideration of boundaries.
   (d) We pay a great deal of attention to it in the initial lectures.

Periodic Wave Equation

1. Next we study the properties of the Periodic Wave Equation

$$\frac{\partial u(x,t)}{\partial t} + a\frac{\partial u(x,t)}{\partial x} = 0 \ , \quad 0 \le x \le 2\pi \tag{2}$$

2. BC: $u(0,t) = u(2\pi,t)$

3. IC: $u(x,0) = g(x), g(0) = g(2\pi)$

Periodic Wave Form

1. The general solution to Eq.1 is:

$$u(x,t) = g(x - at)$$

   with $g(x)$ satisfying the IC

2. We will choose a specific form of the solution for periodic flow

3. Fourier Series: An Arbitrary Periodic (Harmonic) Function Can Be Represented By A Fourier Series

$$g(x) = \sum_{m=-N}^{M} f_m(0) e^{i\kappa_m x} = \sum_m g_m(x) \tag{3}$$

Examples of Periodic Fourier Functions

1. Simple Sine

$$\begin{aligned} sin(x) &= \frac{e^{ix} - e^{-ix}}{2i} \\ M, N &= 1, \kappa_1 = 1, \kappa_{-1} = -1 \\ f_1(0) &= \frac{1}{2i}, \quad f_{-1}(0) = \frac{-1}{2i} \end{aligned}$$

2. Sum of Sine and Cosine

$$2.0 sin(3x) + 0.1 cos(5x) = 2.0\frac{e^{3ix} - e^{-3ix}}{2i} + 0.1\frac{e^{5ix} + e^{-5ix}}{2}$$

$$M, N = 5, \kappa_3 = 3, \kappa_{-3} = -3, \kappa_5 = 5, \kappa_{-5} = -5,$$

$$f_3(0) = \frac{2.0}{2i}, \quad f_{-3} = \frac{-2.0}{2i}, \quad f_5(0) = \frac{0.1}{2}, \quad f_{-5} = \frac{0.1}{2}$$

Linear Superposition Theory

1. Equation 1 is a linear equation in $u(x,t)$ and must satisfy an arbitrary $g(x)$ from Eq.3
2. By the Theory of Linear Superposition, given two or more solutions, e.g., $u_1(x,t), u_2(x,t)$
   (a) If $u_1(x,t)$ Satisfies Eq.1 and $u_2(x,t)$ Satisfies Eq.1
   (b) Then: The sum of $u(x,t) = c_1 u_1(x,t) + c_2 u_2(x,t)$ also satisfies Eq.1, where c_1 and c_2 are arbitrary constants.

Generalize Solution

1. Eq.3 is a sum of various periodic functions $e^{i\kappa_m x}$, each of which taken separately leads to general solutions $u_m(x,t) = g_m(x - at)$
   (a) Simplify and generalize our solutions class by choosing the general $g(x) = e^{i\kappa x}$
   (b) Consider each wave component separately, (ie. general κ)
2. General Solution for Periodic IC

$$u(x,t) = \sum_{m=-N}^{M} f_m(0) e^{i\kappa_m (x-at)} \tag{4}$$

Separation of Variable Solution of Wave Equation

1. Using separation of variables assuming a general form

$$u(x,t) = e^{i\kappa x} f(t)$$

   (arbitrary κ)

2. Apply the general result $\frac{\partial u(x,t)}{\partial x} = i\kappa \, u(x,t)$ to Eq.2,

$$\frac{\partial u(x,t)}{\partial t} + a\frac{\partial u(x,t)}{\partial x} = 0$$

$$\frac{\partial e^{i\kappa x} f(t)}{\partial t} + a i\kappa e^{i\kappa x} f(t) = 0$$

PDE - ODE

1. The ODE for $f(t)$ is
$$\frac{\partial f(t)}{\partial t} + a\ i\kappa\ f(t) = 0$$
with solution
$$f(t) = f(0)e^{-aik\ t}$$
giving
$$u(x,t) = ce^{i\kappa x}e^{-ai\kappa\ t}, \quad c = f(0)$$
2. So the General Solution to Eq.2, (for each κ),
$$u(x,t) = ce^{i\kappa(x-a\ t)} \tag{5}$$

General Solution

$$u(x,t) = \sum_m c_m e^{i\kappa(x - a\ t)} \tag{6}$$

1. This will be the exact solution which we will use to evaluate the effects of
   (a) Approximating $\frac{\partial u}{\partial x}$ with Numerical Finite Differences.
   (b) Approximating $\frac{\partial u}{\partial t}$ with Various Time Advance Schemes.